CHAPTER 1: The Foundations: Logic and Proofs

SECTION 1.1 Propositional Logic
1. Be able to do problems like on the homework. I will give you the list on page 6 of the equivalents to the conditional \( p \rightarrow q \).

SECTION 1.2 Applications of Propositional Logic
1. Be able to convert an English sentence to a statement in propositional logic, such as “If I go to Harry’s or to the country, I will not go shopping.”
   \[ p: \text{I go to Harry’s} \]
   \[ q: \text{I go to the country.} \]
   \[ r: \text{I will go shopping.} \]
   If \( p \) or \( q \) then not \( r \).
   \[ (p \lor q) \rightarrow \lnot r \]

2. Explain the solution to the knight/knave problem.
   An island has two types of inhabitants, knights and knaves.
   Knights always tell the truth.
   Knaves always lie.
   What are \( A \) and \( B \) if:
   (a) \( A \) says “\( B \) is a knight.”
   (b) \( B \) says “The two of us are opposite types.”
   There are two cases we must consider:
   \( A \) is a knight.
   \( A \) is a knave.
   Case 1:
   If \( A \) is a knight, then \( A \) always tells the truth.
   Thus (a) is true, so \( B \) is a knight.
   Thus (b) is true, so they are of opposite types, meaning that \( (p \land \lnot q) \lor (\lnot p \land q) \).
   That means that \( B \) is a knave.
   This is a contradiction because \( B \) can’t be both a knight and a knave. So \( A \) can’t be a knight because it leads to this contradiction.
   Case 2:
   If \( A \) is a knave, then \( A \) always lies.
   Thus (a) is false, so \( B \) is a knave.
   Thus (b) is false, so they are of the same type, further confirming that \( B \) is a knave.
SECTION 1.3 Propositional Equivalencies

1. Define: tautology, contradiction, and contingency.

   A tautology is a proposition which is always true.
   A contradiction is a proposition which is always false.
   A contingency is a proposition which is neither a tautology nor a contradiction.

2. Be able to determine if a proposition is a tautology, a contradiction, or a contingency. For example, \( \neg p \), \( p \lor \neg p \), \( p \land \neg p \).

   Answer: \( \neg p \), is a contingency, \( p \lor \neg p \), is a tautology, and \( p \land \neg p \) is a contradiction as proved below.

   \[
   \begin{array}{c|c|c|c}
   p & \neg p & p \lor \neg p & p \land \neg p \\
   \hline
   T & F & T & F \\
   F & T & T & F \\
   \end{array}
   \]

3. State DeMorgan’s laws. Write the truth tables for them.

   \[
   \neg(p \land q) \equiv \neg p \lor \neg q \\
   \neg(p \lor q) \equiv \neg p \land \neg q
   \]

   \[
   \begin{array}{c|c|c|c|c|c|c|c|c}
   p & q & \neg p & \neg q & (p \land q) & \neg(p \land q) & \neg p \lor \neg q \\
   \hline
   T & T & F & F & T & F & F \\
   T & F & F & T & F & T & T \\
   F & T & T & F & F & T & T \\
   F & F & T & T & F & T & T \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c|c|c|c|c|c|c}
   p & q & \neg p & \neg q & (p \lor q) & \neg(p \lor q) & \neg p \land \neg q \\
   \hline
   T & T & F & F & T & F & F \\
   T & F & F & T & T & T & F \\
   F & T & T & F & T & F & F \\
   F & F & T & T & F & T & T \\
   \end{array}
   \]

4. Be able to write the opposite of an English statement involving \textit{and} or \textit{or} (inclusive) using DeMorgan’s Laws.

   a) Jane is rich and happy. Opposite is Jane is either not rich or not happy or both.

   b) Carlos will bicycle or run tomorrow. Opposite: Carlos will not bicycle and not run tomorrow.
Section 1.4: Predicates and Qualifiers

Be able to do the exercises like those below starting on page 53:

6. Let $N(x)$ be the statement that “$x$ has visited North Dakota,” where the domain consists of the students in your school. Express each of these quantifications in English.
   a) $\exists x N(x)$
   Some student in the school has visited North Dakota. (Alternatively, there exists a student in the school who has visited North Dakota.)
   b) $\forall x N(x)$
   Every student in the school has visited North Dakota. (Alternatively, all students in the school have visited North Dakota.)
   c) $\neg \exists x N(x)$
   This is the negation of part (a): No student in the school has visited North Dakota. (Alternatively, there does not exist a student in the school who has visited North Dakota.)
   d) $\exists x \neg N(x)$
   Some student in the school has not visited North Dakota. (Alternatively, there exists a student in the school who has not visited North Dakota.)
   e) $\neg \forall x N(x)$
   This is the negation of part (b): It is not true that every student in the school has visited North Dakota. (Alternatively, not all students in the school have visited North Dakota.)
   f) $\forall x \neg N(x)$
   All students in the school have not visited North Dakota. (Author: This is technically the correct answer, although common English usage takes this sentence to mean—inaccurately—the answer to part (e). To be perfectly clear, one could say that every student in this school has failed to visit North Dakota, or simply that no student has visited North Dakota.)

8. Translate these statements into English, where $R(x)$ is “$x$ is a rabbit,” and $H(x)$ is “$x$ hops” and the domain consists of all animals.
   a) $\forall x (R(x) \to H(x))$
   If an animal is a rabbit, then that animal hops. (Alternatively, every rabbit hops.)
   b) $\forall x (R(x) \land H(x))$
   Every animal is a rabbit and hops (obviously not true).
   c) $\exists x (R(x) \to H(x))$
   There exists an animal such that if it is a rabbit, then it hops. (Author: Note that this is trivially true, satisfied, for example, by lions, so it is not the sort of thing one would say.)
   d) $\exists x (R(x) \land H(x))$
   There exists an animal that is a rabbit and hops. (Alternatively, some rabbits hop. Alternatively, some hopping animals are rabbits.)

10. Let $C(x)$ be the statement that “$x$ has a cat,” let $D(x)$ be the statement that “$x$ has a dog,” and let $F(x)$ be the statement “$x$ has a ferret.” Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.
   a) A student in your class has a cat, a dog, and a ferret.
   We assume that this means that one student has all three animals: $\exists x (C(x) \land D(x) \land F(x))$.
   b) All students in your class have a cat, a dog, or a ferret.
   $\forall x (C(x) \lor D(x) \lor F(x))$
   c) Some student in your class has a cat and a ferret, but not a dog.
   $\exists x (C(x) \land F(x) \land \neg D(x))$
   d) No student in your class has a cat, a dog, and a ferret.
   This is the negation of part (a): $\neg \exists x (C(x) \land D(x) \land F(x))$.
   e) For each of the three animals, there is a student who has this animal as a pet.
   Here the owners of these pets can be different: $(\exists x C(x)) \land (\exists x D(x)) \land (\exists x F(x))$. There is no harm in using the same dummy variable, but this could also be written, for example, as $(\exists x C(x)) \land (\exists y D(y)) \land (\exists z F(z))$. 